Division by a 2-or-3-Digit Divisor

Pre-requisites

Lots of practice at multiplication, addition and subtraction with the golden beads, stamp game, bead frame and checker board, ideally until able to perform these operations abstractly. The child needs to be very comfortable with multiplication by a two digit multiplier and with the commutative and associative laws of multiplication.

Lots of practice at distributive division by a one digit divisor with the golden beads and stamp game, until very comfortable with it.

An introduction to group division and an understanding of the way group division differs from distributive division.

Materials

Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.
Green division skittles (optional).

Purpose

To introduce division by a two digit divisor, and then by a three digit divisor, using concrete materials (the stamp game) following the same steps that are used when performing division abstractly, so that practice will lead the child to abstraction.

Note

I think of the times symbol ‘x’ as meaning ‘of’ or ‘groups of’. So, for example, I write three groups of six: 3 x 6. If the child you have been working with has been taught to think of the times symbol ‘x’ as meaning ‘multiplied by’ then you will need to write the combination the other way around: 6 x 3 meaning six taken three times.

In some cases, you may also need to use different language to what I use. Where I say, for example, “five groups of 600” you may need to say something like “600 taken five times” or you may wish to say both: “Five groups of 600. 600 taken five times.”
Presentation 1: Revision of the concept of group division

Pre-requisites
Lots of practice at multiplication, addition and subtraction with the golden beads, stamp game, bead frame and checker board, ideally until able to perform these operations abstractly. The child needs to be very comfortable with multiplication by a two digit multiplier and with the commutative and associative laws of multiplication.

Lots of practice at distributive division by a one digit divisor with the golden beads and stamp game until very comfortable with it.

An introduction to group division and an understanding of the way group division differs from distributive division.

Materials
The stamp game.

Purpose
To revise the concept of group division. Therefore if the child has been working on group division recently, this presentation could be skipped.

Presentation
With the child, perform a group division problem with a two digit dividend and a one digit divisor using the stamp game.

E.g. 41 ÷ 3. Change all the tens to ones and arrange all the stamps in groups of three.

Presentation 2 should usually be done immediately.
Presentation 2: One-hierarchy-at-a-time group division

Pre-requisites
Lots of practice at multiplication, addition and subtraction with the golden beads, stamp game, bead frame and checker board, ideally until able to perform these operations abstractly. The child needs to be very comfortable with multiplication by a two digit multiplier and with the commutative and associative laws of multiplication.

Lots of practice at distributive division by a one digit divisor with the golden beads and stamp game until very comfortable with it.

Presentation 1 above or recent practice at group division.

Materials
Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.

Purpose
To introduce “one-hierarchy-at-a-time group division”.

Presentation
With the child, set out a dynamic division problem with the stamp game. The dividend should be four digits and the divisor one digit.
E.g. 6549 ÷ 4

Write the division problem on grid paper in Montessori hierarchical colours:

a b c d f

“We are going to solve this problem using group division.” Depending on the child, you might pretend to start exchanging the six thousands into units. Consider whether the child would find that funny or be horrified at the idea.

“It would take too long to change all these [indicate the quantity] into units and sort them into groups of four so I will sort each hierarchy into groups of four.”
“I can make one group of four thousands.” Place four thousands in a group. Write in the ‘1’ in the thousands column in your quotient:

\[
\begin{array}{cc}
4 & 6 \\
\hline
1 & 5 \\
4 & 9 \\
\end{array}
\]

“There are two thousands left. They need to be exchanged for hundreds.” Make the exchange or have the child do it.

Write a small '2' just to the left of and above the '5' to represent the thousands exchanged for hundreds. Use hierarchical colours if desired (i.e. green in this case) or if you think that will only complicate things for the child, use plain pencil:

\[
\begin{array}{cc}
4 & 6 \\
\hline
1 & 2 \\
5 & 4 \\
9 \\
\end{array}
\]

Arrange the hundreds in groups of four or have the child do it. “There are six groups of four hundreds.” Write the '6' into your answer or have the child do it:

\[
\begin{array}{cc}
4 & 6 \\
\hline
1 & 6 \\
6 & 2 \\
5 & 4 \\
9 \\
\end{array}
\]

Continue in the same way for the tens and ones. Write the final remainder in whatever way the child is used to, e.g. “Remainder 1” or “\(\frac{1}{4}\”).

Have the child do as many examples as they like over the next days.
Presentation 3: One-hierarchy-at-a-time group division by a two digit divisor

Pre-requisites
Presentation 2 above.
Practise at one-hierarchy-at-a-time group division by a one digit divisor till comfortable with the process.

Materials
Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.

Purpose
To introduce one-hierarchy-at-a-time group division by a two digit divisor.

Presentation
With the child, set out a dynamic division problem with the stamp game. The dividend should be four digits and the divisor two digits.
E.g. 3524 ÷ 13
Write the problem on grid paper in Montessori hierarchical colours.
“We are going to solve this problem using group division one-hierarchy-at-a-time.”
“I can't make any groups of 13 with the thousands.”
“I could exchange these three thousands for hundreds but I won't yet. We will just think of each of these thousands as ten hundreds.” Move the thousands to the hundreds.
“I want to make groups of 13 hundreds.”
“Remember each of these thousands is ten hundreds so if I put one thousand and three hundreds in a group, that’s a group of 13 hundreds.” Make the group. Make sure the child is happy with this. If necessary, exchange the thousand for ten hundreds so that the child can see it is a group of 13 hundreds.
Put the second thousand in a group with the remaining two hundreds. “I need one more hundred to make another group of 13 hundreds.” Either elicit from the child the suggestion to exchange the remaining thousand for hundreds or say, “This thousand is ten hundreds. I'll exchange it now so I can take one of its hundreds.” Make the exchange or have the child do it. Complete the group of 13.
“There are only nine hundreds left. I can't make any more groups of 13 hundreds.”
“There are two groups of 13 hundreds.” Write in the ‘2’:

```
2
1 3 3 5 2 4
```

“There are nine hundreds left.”
Write a small ‘9’ just to the left of and above the ‘2’ in the dividend to represent the leftover hundreds.

“We could exchange these for tens but we won't yet. We'll just think of each as ten tens.”
Move the hundreds to the tens.

“I need to make groups of 13 tens.”

“Remember each of these hundreds is ten tens so if I put one hundred and three tens in a group, that would 13 tens.” Put a hundred together with the two tens. “I need one more ten.” Either elicit from the child the suggestion to exchange a hundred for ten tens or say, “This hundred [one of the loose hundreds not the one you have put in a group with the tens] is ten tens. I'll exchange it now so I can put one of its tens in the group.” Make the exchange or have the child do it. Complete the group of 13 tens.

Make three more groups of 13 tens with the loose hundreds and tens.
Put one hundred ready to start another group of 13. Either elicit from the child the suggestion to exchange another loose hundred for ten tens or say, “This hundred [one of the loose hundreds] is ten tens. I'll exchange it now so I can use its tens.” Make the exchange or have the child do it.
Complete the group of 13 tens and make two more groups.

“There is only one ten left. I can't make any more groups of 13 tens.”

“There are seven groups of 13 tens.” Write in the ‘7’:

```
7
1 3 3 5 92 4
```

“There is one left.” Write a small ‘1’ to the left of and above the ‘4’ in the dividend.

“We will think of this as ten ones.” Move the ten to the ones. Put the ten and three ones in a group to make a group of 13 ones.
“There is one group of 13 ones.” Write in the '1':

```
  2  7  1
3  5  9  2  4
1  3
```

“There is one one remaining.” Write the final remainder in whatever way the child is used to, e.g. 'Remainder 1' or '1/13'.

Have the child do at least one more example over the next days then make Presentation 4. The child may need to work the example(s) with you; they are probably not yet ready to work on this alone.
Presentation 4: One-hierarchy-at-a-time group division by a two digit divisor, with estimating

Pre-requisites
Presentation 3 above.
Practice at at least one further example like the one in Presentation 3.

Materials
Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.

Purpose
To facilitate progress towards abstraction of the process.

Presentation
As for Presentation 3 but each step of the way, have the child estimate how many groups they will make. So before beginning to sort the hundreds into groups, ask, “How many groups of 13 [or whatever figure the divisor is this time] hundreds do you think there will be?” Before sorting the tens into groups, ask, “How many groups of 13 tens do you think there will be?” And the same for the ones.

Some children may wish to make an estimate and then check it out by performing the multiplication problem on a separate sheet of grid paper. So, for example, if the divisor is 26 and they estimate there will be three groups of 26 hundreds, they would perform 26 x 3 on a separate sheet. If the product is greater than the number of hundreds available, then clearly the estimate was too high: perhaps there will be two groups of 26 hundreds. If the product is lower than the number of hundreds available, then a subtraction problem can be performed on the separate sheet of paper: take away the product of 26 x 3 from the quantity of hundreds available to see if there are more than 26 remaining, in which case the estimate was too low, or less than 26 remaining, in which case the estimate is correct.

Have the child practise one-hierarchy-at-a-time group division by a two digit divisor, with estimating, over the next days and weeks. This is the path to abstraction.
Presentation 5: Distributive division, identifying the multiplication combinations in each step

Pre-requisites

Presentation 4 above.

Practise at one-hierarchy-at-a-time group division by a two digit divisor, with estimating, till comfortable with the process.

Materials

Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.
Green division skittles (optional).

Purpose

Preparation for Presentation 6, whose purpose is to demonstrate why one-hierarchy-at-a-time group division works to produce the correct answer.

Presentation

Step 1: Revision - Combination families

On a sheet of grid paper, write a basic division combination with answer.

E.g:

\[
\begin{array}{c}
6 \\
4 \underline{2} \underline{4} \\
\end{array}
\]

“I am going to write the other three combinations in this combination family.”

\[
\begin{array}{c}
4 \times 6 = 24 \\
6 \times 4 = 24 \\
4 \\
6 \underline{2} \underline{4} \\
\end{array}
\]

Give the child two or three different basic division combinations and have the child write the other three combinations in the combination family for each.

Be sure to have the child write the division combinations in the form I have above so that
the child makes the connection that the divisor and quotient in a large division problem are factors in a multiplication combination.

**Step 2**

With the child, set out a dynamic division problem with the stamp game. The dividend should be four digits and the divisor one.

Write the problem on grid paper in Montessori hierarchical colours. E.g:

```
4  2  5  4  3
```

“We are going to use distributive division to solve this problem.” Lay out the four skittles. Exchange or have the child exchange the thousands and distribute the hundreds. “Each of the four piles has six hundreds.” On a separate sheet of grid paper write:

4 x 600 (or 600 x 4)

Write the ‘6’ into the quotient:

```
6
4  2  5  4  3
```

Point out that the ‘6’ is in the hundreds column because it represents 600. Exchange or have the child exchange the leftover hundred for tens and distribute. “Each of the four piles has three tens.”

Write on the separate sheet of grid paper:

4 x 30 (or 30 x 4)

Write the ‘3’ into the quotient, pointing out that it is in the tens column because it represents three tens:

```
6  3
4  2  5  4  3
```

Exchange or have the child exchange the leftover tens for ones and distribute. “Each pile has five ones.”

Write on the separate sheet:

4 x 5 (or 5 x 4)
Write the '5' into the quotient, pointing out that it is in the ones column because it represents five ones:

\[
\begin{array}{c|cccc}
4 & 6 & 3 & 5 \\
\hline
2 & 5 & 4 & 3 \\
\end{array}
\]

Record the remainder in whatever way the child is familiar with: 'Remainder 1' or '¾'.

Do Presentation 6 immediately or within the next few days.
Presentation 6: One-hierarchy-at-a-time group division, identifying the multiplication combinations in each step

Pre-requisites
Presentation 5 above.

Materials
Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.

Purpose
To demonstrate why one-hierarchy-at-a-time group division works to produce the correct answer.

Presentation
Step 1: Revision of the commutative and associative laws of multiplication
Use any activity the child would enjoy to remind the child that 2 x 300 is the same as 3 x 200 etc.
Some children might enjoy breaking down and rebuilding combinations on paper using the associative and commutative laws:
2 x 300 = 2 x (3 x 100) = 2 x 3 x 100 = 3 x 2 x 100 = 3 x (2 x 100) = 3 x 200
Some children might enjoy pairing equal combinations, either written on cards or built with the stamp game.

Step 2
With the child, set out the same dynamic division problem performed in presentation 5 with the stamp game.
Write the problem on grid paper in Montessori hierarchical colours. E.g:

```
4 2 5 4 3
```

“We are going to use one-hierarchy-at-a-time group division to solve this problem.”
Exchange or have the child exchange the thousands and then make groups of four hundreds.
“There are six groups of four hundred” On a separate sheet of grid paper write:
6 x 400  (or 400 x 6)

“Six groups of 400 is the same as 600 groups of 4.” Write:
= 600 x 4 (or 4 x 600)

Write the ’6‘ into the quotient:

\[
\begin{array}{c|cccc}
   & 6 & 2 & 5 & 4 & 3 \\
\hline
4 & & & & & \\
\end{array}
\]

Point out that the ’6‘ is in the hundreds column because four fits into 2500 600 times. It may be necessary to discuss this further: “Four hundred fits into 2500 six times so four fits into 2500 600 times.” Refer to the groups laid out and the equations written on the separate sheet.

Exchange or have the child exchange the leftover hundred for tens and make groups of four tens.

“There are three groups of four tens.”

Write on the separate sheet of grid paper:
3 x 40 (or 40 x 3)

”Three times 40 is the same as 30 times four.” Write:
30 x 4 (or 4 x 30)

Write the ’3‘ into the answer, pointing out that it is in the tens column because four fits into 14 tens thirty times. Once again you may need to discuss this further: “Forty fits into 14 tens three times so four fits into 14 tens 30 times.” Refer to the groups laid out and the equations written on the separate sheet.

\[
\begin{array}{c|cccc}
   & 6 & 3 & & & \\
\hline
4 & & & 2 & 5 & 4 & 3 \\
\end{array}
\]

Exchange or have the child exchange the leftover tens for ones and make groups of four ones.

“There are five groups of four.”

Write on the separate sheet:
5 x 4 (or 4 x 5)
Write the '5' into the answer, pointing out that four fits into 23 five times.

\[
\begin{array}{c|cccc}
& 6 & 4 & 5 \\
\hline
4 & 2 & 5 & 4 & 3 \\
\end{array}
\]

Record the remainder in whatever way the child is familiar with: 'Remainder 3' or '⅓'.
Do further examples over the next days if desired.
Presentation 7: One-hierarchy-at-a-time group division by a three digit divisor, with estimating

Pre-requisites
Presentation 4 above.
Practise at one-hierarchy-at-a-time group division by a two digit divisor, with estimating, till comfortable with the process.

Materials
Grid paper.
Pencil.
Green, blue, red coloured pencils.
The stamp game.

Purpose
To introduce division by a three digit divisor, using concrete materials to follow the same steps that are used when performing division abstractly, so that practice will lead the child to abstraction.

Presentation
As for Presentation 4 but create an example with a three digit divisor.

Encourage the child to practise one-hierarchy-at-a-time group division, with estimating, by one digit, two digit and three digit divisors as much as desired. Abstraction of the process should come about through practice.

At this point, there may be a break from division presentations for a long period. Presentation 8 should be made shortly before the child begins to need to carry out very large division problems (e.g. 123,456,789,123 ÷ 9,876,543) or division of polynomials. Presentation 8 uses the same method for solving division problems as described in Presentations 4 and 7 but more of the steps are recorded on paper. This recording makes it feasible to solve more complex problems.
Presentation 8: One-hierarchy-at-a-time group division, with recording of the steps

Pre-requisites
Presentation 4 above.
Practice till comfortably able to perform division by a two or three digit divisor abstractly.

Materials
Grid paper.
Pencil.

Purpose
To introduce recording of the steps of one-hierarchy-at-a-time group division so that the child learns the tools necessary for carrying out more complex division problems such as division involving very large numbers and division of polynomials.

Presentation
On grid paper, write a dynamic division problem with a two digit divisor, e.g:

\[
\begin{array}{c}
2 & 3 & 8 & 0 & 4 & 1 \\
\end{array}
\]

Have the child carry out the first step of the problem: working out how many groups of 23 hundreds are in 80 hundreds. When the child identifies that there are three groups, write the '3' into the quotient:

\[
\begin{array}{c}
2 & 3 & 8 & 0 & 4 & 1 \\
3 \\
\end{array}
\]

The child will perform the subtraction 80 – 69 in order to work out how many hundreds are left to carry over to the tens. If the child goes to carry it out on paper, say “We will carry out the subtraction in a special place this time.” If the child carries it out in their head, say, “We will record the subtraction this time – here.”
Write in the '69' and carry out the subtraction:

```
  3
 2 3 8 0 4 1
  6 9
  1 1
```

“And of course there are four tens and a one left to divide as well.” Write in:

```
  3
 2 3 8 0 4 1
  6 9
  1 1 4 1
```

Have the child carry out the next step: working out how many groups of 23 tens are in 114 tens. When the child identifies that there are four groups, write the '4' into the quotient:

```
  3 4
 2 3 8 0 4 1
  6 9
  1 1 4 1
```

When the child goes to work out how many tens are still remaining, as before show them to work it out on the problem:

```
  3 4
 2 3 8 0 4 1
  6 9
  1 1 4 1
  9 2
  2 2 1
```
Have the child carry out the next step: working out how many groups of 23 there are in 221. When the child identifies that there are nine groups, write the '9' into the quotient:

```
   3  4  9
2 3 8 0 4 1
  6 9
 1 1 4 1
  9 2
 2 2 1
```

When the child goes to work out how many ones are still remaining, as before show them to work it out on the problem:

```
   3  4  9
2 3 8 0 4 1
  6 9
 1 1 4 1
  9 2
 2 2 1
 2 0 7
  8
```

Re-write the above with estimating and having to change the quotient.
The child may wish to record the remainder 'Remainder 8' or '8/23' or may wish to continue dividing for a few decimal places.

Some children may enjoy practising with similar examples over the next days and weeks. Other children may see no point: they can solve the problem without recording all the steps, why bother recording them. In that case, give them problems to solve that are impractical to solve without recording steps, such as:

```
123,456,789,123 ÷ 9,876,543
```

Make sure they work on grid paper; it's just too easy to make mistakes otherwise when working with such large numbers.
They will probably want to check their answers with a calculator and go through the problem again if there is an error.

This method can be used to solve division of polynomial problems.